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Large matchings from eigenvalues

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Abstract

We find lower bounds on the difference between the spectral radius λ_1 and the average degree $\frac{2e}{n}$ of an irregular graph G of order n and size e. In particular, we show that, if $n \geqslant 4$, then

$$\lambda_1 - \frac{2e}{n} > \frac{1}{n(\Delta + 2)}$$

where Δ is the maximum of the vertex degrees in G.

Brouwer and Haemers found eigenvalue conditions sufficient to imply the existence of perfect matchings in regular graphs. Using the above bound, we refine and extend their results to obtain sufficient conditions for the existence of large matchings in regular graphs.

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1. Preliminaries

Throughout the paper, G denotes a simple graph with edge set E, vertex set $V = [n] = \{1, 2, \ldots, n\}$ and corresponding degrees $d_1, d_2, \ldots, d_n$. The maximum and minimum degrees

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of G are denoted by Δ and δ, respectively. The terms *order* and *size* refer to the numbers $n = |V|$ of vertices and $e = |E|$ of edges of G, respectively. The *eigenvalues* of G are the eigenvalues λ_i of its adjacency matrix A, indexed so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$.

One of the first results in spectral graph theory, due to Collatz and Sinogowitz [4], states that for any graph with n vertices and e edges,

$$\lambda_1 \geqslant \frac{2e}{n} \tag{1}$$

with equality if and only if the graph is regular.

In the next two sections, we find lower bounds for $\lambda_1 - \frac{2e}{n}$ when G is irregular. In particular, our results will imply that, for all graphs G on $n \geqslant 4$ vertices,

$$\lambda_1 - \frac{2e}{n} > \frac{1}{n(\Delta + 2)}. \tag{2}$$

We require this bound in the last two sections to obtain eigenvalue conditions sufficient to imply the existence of large matchings in regular graphs.

2. Spectral radius and average degree

Using a Rayleigh–Ritz ratio [7, p. 176], we first note that if x is a unit column vector in $\mathbb{R}^n$, then $x^{\mathrm{t}}x = 1$ and

$$\lambda_1 \geqslant x^{\mathrm{t}}Ax = 2\sum_{ij\in E} x_i x_j \tag{3}$$

with equality if and only if x is an eigenvector corresponding to λ_1. For example, choosing $x_i = \frac{1}{\sqrt{n}}$ gives inequality (1).

We first obtain a lower bound on $\lambda_1 - \frac{2e}{n}$ in terms of the restriction $d_S = (d_i)_{i\in S}$ of the degree sequence of G to a subset S of the vertex set.

Theorem 1. *Let S be a set of s vertices in G. Then*

$$\lambda_1 - \frac{2e}{n} \geqslant \frac{1}{cn}\left(\frac{\sum_{i,j\in S}(d_i - d_j)^2}{\sqrt{s\sum_{i\in S}d_i^2} + \sum_{i\in S}d_i}\right), \tag{4}$$

where $c = 1$ if the vertices in S are independent and $c = 2$ otherwise.

Proof. Let $s = |S|$ denote the number of vertices in S. Taking $x_i = \frac{a_i}{\sqrt{n}}$ for $i \in S$ and $x_i = \frac{1}{\sqrt{n}}$ when $i \notin S$, we have $||x|| = 1$ when $\sum_{i\in S} a_i^2 = s$. If the vertices of S are independent, then

$$\lambda_1 - \frac{2e}{n} \geqslant 2\sum_{ij\in E} x_i x_j - \frac{2e}{n} = \frac{2}{n}\left(\sum_{i\in S} a_i d_i - \sum_{i\in S} d_i\right). \tag{5}$$

If we choose the a_i so that equality holds in the Cauchy–Schwartz inequality, $\left(\sum_{i\in S} a_i d_i\right)^2 \leqslant s\left(\sum_{i\in S} d_i^2\right)$, then

$$\lambda_1 - \frac{2e}{n} \geqslant \frac{2}{n}\left(\sqrt{s\sum_{i\in S} d_i^2} - \sum_{i\in S} d_i\right) = \frac{1}{n}\left(\frac{\sum_{i,j\in S}(d_i - d_j)^2}{\sqrt{s\sum_{i\in S}d_i^2} + \sum_{i\in S}d_i}\right),$$

where the last equality follows from the Lagrange identity.

Suppose now that the set S is not independent in G. Consider the bipartite double cover $\widetilde{G}$ of G. The graph $\widetilde{G}$ has vertex set $V(G) \times \{1, 2\}$ with $(i, a) \sim (j, b)$ if $i \sim j$ in G and $a \neq b$. Thus, the biadjacency matrix of $\widetilde{G}$ is the adjacency matrix of G and it is easy to see that $\lambda_1(G) = \lambda_1(\widetilde{G})$ and that G and $\widetilde{G}$ have the same average degree. The set $S \times \{1\}$ is an independent set of size s in $\widetilde{G}$. Because the number of vertices has doubled, by applying the previous inequality, we obtain the case with $c = 2$. $\square$

The bound (4) may be written compactly in terms of the first two moments, $m_r(d_S) = \frac{1}{n}\sum_{i\in S} d_i^r$, $r = 1, 2$, and the variation, $\mathrm{var}(d_S) = m_2(d_S) - m_1^2(d_S)$, of the restricted degree sequence $d_S = (d_i)_{i\in S}$

$$\lambda_1 - \frac{2e}{n} \geqslant \frac{2s}{cn}\left(\frac{\mathrm{var}(d_S)}{\sqrt{m_2(d_S)} + m_1(d_S)}\right). \tag{6}$$

Taking S to be a pair of vertices with distinct degrees, we obtain the following corollary to Theorem 1.

Corollary 2. *Suppose i and j are vertices of G and $d_i > d_j$. Then*

$$\lambda_1 - \frac{2e}{n} \geqslant \frac{2(d_i - d_j)^2}{cn\left(\sqrt{2(d_i^2 + d_j^2)} + (d_i + d_j)\right)} > \frac{(d_i - d_j)^2}{2cnd_i},$$

where $c = 1$ if i and j are not adjacent and $c = 2$ if i and j are adjacent.

The next result is an immediate consequence of Corollary 2.

Corollary 3. *For every graph G,*

$$\lambda_1 - \frac{2e}{n} \geqslant \frac{(\Delta - \delta)^2}{4n\Delta}.$$

The previous corollary implies that the required bound (2) holds for an irregular graph G except possibly when $\Delta - \delta = 1$. In the next section, we find that (2) holds for these remaining cases.

3. Graphs with $\delta = \Delta - 1$

Suppose that $V = V_1 \cup V_2$ is a partition of the vertex set V of G. For $i = 1, 2$, let G_i be the subgraph of G induced by V_i, and let $n_i = |V_i|$ be the number of vertices in G_i and e_i the number of edges. Also, let G_{12} be the bipartite subgraph induced by the partition and let e_{12} be the number of edges in G_{12}. A theorem of Haemers [6] asserts that the eigenvalues of the quotient matrix of the partition interlace the eigenvalues of the adjacency matrix of G. Applying this result to the maximum eigenvalues, it turns out that

$$\lambda_1 - \frac{2e}{n} \geqslant \frac{e_1}{n_1} + \frac{e_2}{n_2} + \sqrt{\left(\frac{e_1}{n_1} - \frac{e_2}{n_2}\right)^2 + \frac{e_{12}^2}{n_1 n_2}} - \frac{2e}{n}$$

with equality if and only if G_1 and G_2 are regular and G_{12} is semiregular.

For the special case where $\delta = \Delta - 1$ in G, let V_1 be the set of vertices of degree Δ, and V_2 the set of vertices of degree $\Delta - 1$. Let $\epsilon = \frac{e_{12}}{n_1 n_2}$. Then, $0 \leqslant \epsilon \leqslant 1$. Noting that $n = n_1 + n_2$, $n_1\Delta = 2e_1 + e_{12}$, and $n_2(\Delta - 1) = 2e_2 + e_{12}$, the inequality above can be shown to reduce to

$$\lambda_1 - \frac{2e}{n} \geqslant \frac{1}{2}\sqrt{(\epsilon n-1)^2 + 4\epsilon n_1} - \frac{1}{2}(\epsilon n - 1) - \frac{n_1}{n} \tag{7}$$

with equality if and only if G_1 and G_2 are regular.

We have $\lambda_1 - \frac{2e}{n} \geqslant \frac{1}{2}(b-a)$, where $b = \sqrt{(\epsilon n-1)^2 + 4\epsilon n_1}$ and $a = \epsilon n - 1 + \frac{2n_1}{n}$. Because G is irregular, $b > a$ and it follows that:

$$\lambda_1 - \frac{2e}{n} \geqslant \frac{b^2-a^2}{2(b+a)} > \frac{b^2-a^2}{4b} = \frac{n_1 n_2}{n^2\sqrt{(\epsilon n-1)^2+4\epsilon n_1}} > \frac{n_1 n_2}{n^2(\epsilon n+1)}. \tag{8}$$

Together with Corollary 3, the following theorem implies that the required bound (2) holds for all graphs of order $n \geqslant 4$.

Theorem 4. *If G is a graph on $n \geqslant 4$ vertices with $\delta = \Delta - 1$, then*

$$\lambda_1 - \frac{2e}{n} > \frac{1}{n(\Delta+2)}. \tag{9}$$

The proof is rather long and is given below by a sequence of lemmas which, in some cases, provide finer bounds than the one in Theorem 4. Throughout this section, we always assume that G is a graph of order $n \geqslant 4$ with n_1 vertices of degree Δ and $n_2 = n - n_1$ vertices of degree $\delta = \Delta - 1$. As above, $\epsilon = \frac{e_{12}}{n_1 n_2}$ where e_{12} denotes the number of edges with ends of different degrees. We may also assume that $\Delta \geqslant 2$. For if $\Delta = 1$ then G consists of isolated edges and vertices, $\lambda_1 = 1$, and it is easily checked that Theorem 4 holds. The assumption that $n \geqslant 4$ is needed because the theorem fails when G is a path on three vertices.

Lemma 5

$$\lambda_1 - \frac{2e}{n} > \frac{1}{n^2}\max\left\{\frac{n_1 n_2^2}{n\Delta + n_2}, \frac{n_1^2 n_2}{n\Delta - n_2}\right\}. \tag{10}$$

Proof. Because $e_{12} \leqslant n_1\Delta$ and $n_2(\Delta-1)$, we have $\epsilon \leqslant \frac{\Delta}{n_2}$ and $\frac{\Delta-1}{n_1}$. Substituting these simple bounds in (8) gives (10). □

Corollary 6. *Let G be a graph of order $n \geqslant 4$ with $\delta = \Delta - 1$. Then*

$$\lambda_1 - \frac{2e}{n} > \begin{cases} \frac{1}{n\Delta+n_2} & \text{if } 4 \leqslant n_1 \leqslant \frac{n}{2}, \\ \frac{1}{n\Delta-n_2} & \text{if } 4 \leqslant n_2 \leqslant \frac{n}{2}. \end{cases}$$

Proof. If $4 \leqslant n_1 \leqslant \frac{n}{2}$, then $n_2 \geqslant \frac{n}{2}$, so $n_1 n_2^2 \geqslant n^2$ and the first inequality follows from Lemma 5. The proof of the second inequality is similar. □

Because the corollary implies that Theorem 4 holds whenever the conditions $n_1 \geqslant 4$ and $n_2 \geqslant 4$ both hold, it remains only to consider the cases where $n_1 \leqslant 3$ and those where $n_2 \leqslant 3$.

Lemma 7. *If $n \geqslant 4$ and $n_2 \leqslant 3$, then $\lambda_1 - \frac{2e}{n} > \frac{1}{n(\Delta+2)}$.*

Proof. If $n_2 = 1$, then by (10), it is sufficient to show that $\frac{(n-1)^2}{n^2(n\Delta-1)} \geqslant \frac{1}{n(\Delta+2)}$ or $(n-1)^2(\Delta+2) \geqslant n(n\Delta-1)$ or $2n^2 \geqslant (2n-1)\Delta + 3n - 2$. Because $\Delta \leqslant n-1$, the latter always holds.

If $n_2 = 2$, then by (10), it is enough to have $\frac{2(n-2)^2}{n^2(n\Delta-2)} \geqslant \frac{1}{n(\Delta+2)}$ or $2(n-2)^2 \geqslant \frac{n^2\Delta}{\Delta+2} - \frac{2n}{\Delta+2}$. Because $2 \leqslant \Delta \leqslant n-1$, it follows that $\frac{\Delta}{\Delta+2} \leqslant \frac{n-1}{n+1}$ and $\frac{2n}{\Delta+2} \geqslant \frac{n}{2}$. Thus, it is sufficient that $2(n-2)^2 \geqslant \frac{n^2(n-1)}{n+1} - \frac{n}{2}$. This holds for $n \geqslant 4$.

If $n_2 = 3$, then by (10), it is sufficient to show that $\frac{3(n-3)^2}{n^3\Delta} \geqslant \frac{1}{n(\Delta+2)}$ or $3(n-3)^2 \geqslant n^2\frac{\Delta}{\Delta+2}$. Because $\frac{\Delta}{\Delta+2} \leqslant \frac{n-1}{n+1}$, it is enough to have $3(n-3)^2 \geqslant \frac{n^2(n-1)}{n+1}$. The latter holds whenever $n \geqslant 6$. There is no graph with $n_2 = 3$ when $n = 4$. When $n_2 = 3$ and $n = 5$, we have a complete bipartite graph and the lemma is easily seen to hold. □

It now remains to show that Theorem 4 holds whenever $n_1 \leqslant 3$ and $n_2 \geqslant 4$.

Lemma 8. *If $n_1 \leqslant 3$ and $n_2 \geqslant 4$ then $\lambda_1 - \frac{2e}{n} > \frac{1}{n(\Delta+2)}$.*

Proof. If $n_1 = 1$, then $\epsilon = \frac{e_{12}}{n_1 n_2} = \frac{\Delta}{n-1}$. By (8), it is sufficient to show that

$$\frac{n_1 n_2}{n^2\sqrt{\left(\frac{n\Delta}{n-1} - 1\right)^2 + \frac{4\Delta}{n-1}}} = \frac{(n-1)^2}{n^2\sqrt{((\Delta-1)n+1)^2 + 4\Delta(n-1)}} > \frac{1}{n(\Delta+2)}$$

or

$$(n-1)^2(\Delta+2) \geqslant n\sqrt{((\Delta-1)n+1)^2 + 4\Delta(n-1)}.$$

Because $\sqrt{x^2+a} \leqslant x + \frac{a}{2x}$, it is enough to have

$$(n-1)^2(\Delta+2) \geqslant (\Delta-1)n^2 + n + \frac{2\Delta(n-1)n}{(\Delta-1)n+1}, \quad \text{where}$$

$$\frac{2\Delta(n-1)n}{(\Delta-1)n+1} < \frac{2\Delta(n-1)}{\Delta-1} \leqslant 3(n-1) \quad \text{when } \Delta \geqslant 3.$$

(We may assume that $\Delta \geqslant 3$ in this case, because, if $\Delta = 2$, then the graph consists of a 3-path together with one or more isolated edges and it is easy to check that Theorem 4 holds then.) Thus, it is sufficient to show that

$$(n-1)^2(\Delta+2) \geqslant (\Delta-1)n^2 + n + 3(n-1), \quad \text{or}$$

$$3n^2 \geqslant (\Delta+2)(2n-1) + n + 3(n-1).$$

Since $\Delta \leqslant n-1$, it is sufficient to have $3n^2 \geqslant (n+1)(2n-1) + 4n - 3$ or $n^2 \geqslant 5n-4$ or $n \geqslant 4$, so this case is completed.

If $n_1 = 2$, then by (10), it is sufficient to show that $\frac{2(n-2)^2}{n^3(\Delta+1)} \geqslant \frac{1}{n(\Delta+2)}$. But $\frac{\Delta+1}{\Delta+2} \leqslant \frac{n}{n+1}$, so it is enough to have $2(n-2)^2 \geqslant \frac{n^3}{n+1}$. This holds for $n \geqslant 6$, which is the case here since $n_2 \geqslant 4$ is assumed.

If $n_1 = 3$, then by (10), it is enough to show that $\frac{3(n-3)^2}{n^3(\Delta+1)} \geqslant \frac{1}{n(\Delta+2)}$. Because $\frac{\Delta+1}{\Delta+2} \leqslant \frac{n}{n+1}$, it is sufficient to show that $3(n-3)^2 \geqslant \frac{n^3}{n+1}$. This holds for $n \geqslant 7$, which is the case here. □

The proof of Theorem 4 is now complete. The lower bound there cannot be increased to $\frac{1}{n(\Delta+1)}$ because a careful analysis for the cocktail party graph (the complement of a maximum matching in a complete graph) of odd order n gives $\lambda_1 - \frac{2e}{n} < \frac{1}{n^2} = \frac{1}{n(\Delta+1)}$.

The bound in the next lemma provides finer bounds than Theorem 4 for some graphs with $\delta = \Delta - 1$ and $\Delta \leqslant 5$. It will be required in the next two sections for results on matchings.

Lemma 9. *Suppose that G is a graph with $\delta = \Delta - 1$, and that G has three vertices u, v, w where u and v are adjacent and of degree Δ, and w is adjacent to neither u nor v and has degree $\Delta - 1$. Then*

$$\lambda_1 - \frac{2e}{n} \geqslant \frac{2}{n}(a^2 + (2a+b)(\Delta - 1) - 3\Delta + 2),$$

where $a = \frac{\sqrt{3}\Delta}{\sqrt{3\Delta^2 - 2\Delta + 1}}$ *and* $b = \frac{\sqrt{3}(\Delta-1)}{\sqrt{3\Delta^2 - 2\Delta + 1}}$.

Proof. Assign to u and v the weight $\frac{a}{\sqrt{n}}$, and to w the weight $\frac{b}{\sqrt{n}}$. Give each remaining vertex the weight $\frac{1}{\sqrt{n}}$. These weights yield a unit vector x so, substituting in the inequality in (3), we get

$$\begin{aligned}\lambda_1 - \frac{2e}{n} &\geqslant \frac{2}{n}(a^2 + 2a(\Delta - 1) + b(\Delta - 1) + (e - 3\Delta + 2)) - \frac{2e}{n} \\ &= \frac{2}{n}(a^2 + (2a+b)(\Delta - 1) - 3\Delta + 2). \qquad \square\end{aligned}$$

For example, for $\Delta = 3, 4, 5$ we get $\lambda_1 - \frac{2e}{n} > \frac{0.2713}{n}, \frac{0.1945}{n}, \frac{0.1512}{n}$, respectively and each of these bounds is greater than $\frac{1}{n(\Delta+2)}$.

4. Matchings and eigenvalues

Recently, there have been a number of results relating the spectrum of a graph G to the existence of certain subgraphs of G. We refer the reader to the survey [8]. Brouwer and Haemers proved the following result in [2].

Theorem 10. *A connected k-regular graph on n vertices with eigenvalues $k = \lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$, and n even which satisfies*

$$k - \lambda_3 \geqslant \begin{cases} 1 - \frac{3}{k+1} & \text{if } k \text{ is even}, \\ 1 - \frac{3}{k+2} & \text{if } k \text{ is odd}, \end{cases} \tag{11}$$

has a perfect matching.

For k odd, Theorem 10 was improved in [3] where it was shown that if $k - \lambda_3 \geqslant 1 - \frac{4}{k+2}$, then G has a perfect matching.

The matching number of the graph G, denoted by $\nu(G)$, is the size of a maximum matching of G. If the number n of vertices of G is even, then $\nu(G) = \frac{n}{2}$ is equivalent to G having a perfect matching.

Using the lower bound (2), we can further improve Theorem 10 as follows.

Theorem 11. *Let G be a connected k-regular graph on n vertices with eigenvalues $k = \lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$. Assume $r > 0$ is an integer such that $n \equiv r$ (mod 2). If*

$$k - \lambda_{r+1} > \begin{cases} 0.1457 & \textit{if } k = 3, \\ 1 - \frac{3}{k+1} - \frac{1}{(k+1)(k+2)} & \textit{if } k \textit{ is even}, \\ 1 - \frac{4}{k+2} - \frac{1}{(k+2)^2} & \textit{if } k \geqslant 5 \textit{ is odd}, \end{cases} \tag{12}$$

then $\nu(G) \geqslant \frac{n-r}{2} + 1$.

Proof. The proof is similar to that in [2] which uses Tutte's theorem on perfect matchings [9, p. 137]. Here we use the Berge–Tutte formula which asserts (see [1] or [9, p. 139]) that

$$\nu(G) = \frac{1}{2}\left(n + \min_{S \subset V(G)} (|S| - \mathrm{odd}(G \setminus S))\right), \tag{13}$$

where $\mathrm{odd}(G \setminus S)$ denotes the number of odd components of $G \setminus S$.

Suppose that $\nu(G) \leqslant \frac{n-r}{2}$. By the Berge–Tutte formula, it follows that there is a subset S such that $q = \mathrm{odd}(G \setminus S)$, $|S| = s$ and $q \geqslant s + r$.

Let $G_1, \ldots, G_q$ denote the odd components of $G \setminus S$. Denote by n_i and e_i the order and the size of G_i, respectively.

For $i \in [q]$, denote by t_i the number of edges with one endpoint in G_i and the other in S. Because G is connected, it follows that $t_i \geqslant 1$ for each $i \in [q]$. Also, since vertices in G_i are adjacent only to vertices in G_i or S, we deduce that $2e_i = kn_i - t_i = k(n_i - 1) + k - t_i$. Because n_i is odd, it follows that $k - t_i$ is even. Thus, t_i has the same parity as k for each $i \in [q]$.

The sum of the degrees of the vertices in S is at least the number of edges between S and $\cup_{i=1}^{q} G_i$. Thus, $ks \geqslant \sum_{i=1}^{q} t_i$. Since $q \geqslant s + r$, it follows that there are at least $r + 1$ t_i's such that $t_i < k$. This implies there are at least $r + 1$ t_i's satisfying $t_i \leqslant k - 2$. We may assume that the t_i are indexed so that $t_i \leqslant k - 2$ for $i \in [r + 1]$. If $t_i \leqslant k - 2$, then $n_i > 1$ and $n_i(n_i - 1) \geqslant 2e_i = kn_i - t_i \geqslant kn_i - k + 2$. Thus, $n_i \geqslant k + \frac{2}{n_i - 1}$, and so $n_i \geqslant k + 1$. If k is odd, then we obtain $n_i \geqslant k + 2$ since n_i is also odd.

If $k = 3$, then $t_i = 1$, $n_i \geqslant 5$ and $2e_i = 3n_i - 1$ for $i \in [r + 1]$. Thus, for each $i \in [r + 1]$, the graph G_i has exactly one vertex u_i of degree 2 and $n_i - 1 \geqslant 4$ vertices of degree 3. Thus, there are two adjacent vertices of degree 3 in G_i which are not adjacent to u_i. The average degree of G_i is $\frac{3n_i - 1}{n_i} = 3 - \frac{1}{n_i}$. From Lemma 9, for each $i \in [r + 1]$, we get

$$\lambda_1(G_i) > 3 - \frac{1}{n_i} + \frac{0.2713}{n_i} = 3 - \frac{0.7287}{n_i} > 2.8543.$$

Suppose now that $k > 3$. We recall that $2e_i = kn_i - t_i \geqslant kn_i - k + 2$ and $n_i > k$ for each $i \in [r + 1]$. This implies that $\frac{2e_i}{n_i} \geqslant k - \frac{k-2}{n_i}$ and thus that each G_i is irregular for $i \in [r + 1]$. If k is even, then $n_i \geqslant k + 1$ and (2) implies that

$$\lambda_1(G_i) \geqslant \frac{2e_i}{n_i} + \frac{1}{n_i(k+2)} \geqslant k - \frac{k-2}{n_i} + \frac{1}{n_i(k+2)} \geqslant k - \frac{k-2}{k+1} + \frac{1}{(k+1)(k+2)}$$

for each $i \in [r + 1]$. Similarly, if k is odd, $n_i \geqslant k + 2$ for each $i \in [r + 1]$ and

$$\lambda_1(G_i) \geqslant k - \frac{k-2}{k+2} + \frac{1}{(k+2)^2}.$$

Using interlacing for the eigenvalues of $G_1 \cup \cdots \cup G_{r+1}$ and G, we obtain

$$\lambda_{r+1}(G) \geqslant \max_{i \in [r+1]} \lambda_1(G_i)$$

The theorem now follows from the results of the previous three paragraphs. $\square$

When $r = 2$, we obtain the following improvement of Theorem 10.

Corollary 12. *A connected k-regular graph on n vertices with eigenvalues $k = \lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$, and n even which satisfies*

$$k - \lambda_3 > \begin{cases} 0.1457 & \text{if } k = 3, \\ 1 - \frac{3}{k+1} - \frac{1}{(k+1)(k+2)} & \text{if } k \text{ is even}, \\ 1 - \frac{4}{k+2} - \frac{1}{(k+2)^2} & \text{if } k \geqslant 5 \text{ is odd}, \end{cases} \tag{14}$$

has a perfect matching.

The results of Theorem 10 and Corollary 12 are best possible up to order $O(k^{-2})$. For there are examples of k-regular graphs with no perfect matchings having $k - \lambda_3 = 1 - \frac{3}{k+1} + O(k^{-2})$ when k is even and $k - \lambda_3 = 1 - \frac{4}{k+2} + O(k^{-2})$ when k is odd. We describe such graphs below.

For k even, Brouwer and Haemers [2] construct examples of k-regular graphs with

$$\lambda_3 = \frac{k - 2 + \sqrt{k^2 + 12}}{2} = k - 1 + \frac{3}{k+1} + O(k^{-2})$$

that contain no perfect matchings. This is done by taking k copies of G', where G' is obtained from K_{k+1} by deleting a matching of size $\frac{k-2}{2}$. Now add $k - 2$ new vertices and join each of these vertices to a vertex of degree $k - 1$ in each G'.

For k odd and $k \geqslant 5$, the following construction appears in [3]. Let H' be a graph on $k + 2$ vertices whose complement is the union of two disjoint edges and a cycle of length $k - 2$. Take k copies of H'. Add $k - 2$ new vertices and join each of these vertices to a vertex of degree $k - 1$ in each H'. This is a connected k-regular graph that has $k^2 + 3k - 2$ vertices and no perfect matchings. Also,

$$\lambda_3 = \lambda_1(H') = \frac{k - 3 + \sqrt{k^2 + 2k + 17}}{2} = k - 1 + \frac{4}{k+2} + O(k^{-2}).$$

However, for small values of k, the improvements from Corollary 12 are significant. Theorem 10 shows that a 3-regular graph with $\lambda_3 \leqslant 2.8$ has a perfect matching. Corollary 12 implies that every 3-regular graphs with $\lambda_3 < 2.8543$ has a perfect matching. The 3-regular graph in Fig. 1 has no perfect matchings and $\lambda_3 = 2.8558$.

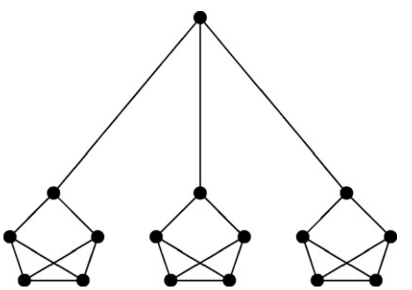

Fig. 1. A 3-regular graph without perfect matchings. Here, $\lambda_3 = 2.8558$.

5. Factor-critical graphs and eigenvalues

A graph G is factor-critical if for each $x \in V(G)$, the subgraph $G \setminus \{x\}$ has a 1-factor. This is a stronger property than $\nu(G) = \frac{n-1}{2}$. Gallai [5] (see also [9], Exercise 3.3.25, p. 147) proved that G is factor critical if and only if $|V(G)|$ is odd and

$$\operatorname{odd}(G \setminus S) \leqslant |S| \text{ for each } S \subset V(G). \tag{15}$$

Theorem 13. *Let G be a k-regular graph on n vertices, n odd. If*

$$k - \lambda_2 > 1 - \frac{3}{k+1} - \frac{1}{(k+1)(k+2)}$$

then G is factor-critical.

Proof. Suppose that G satisfies the conditions of the theorem and G is not factor-critical. By Gallai's condition (15), there is a subset $S \subset V(G)$ such that $q = \operatorname{odd}(G \setminus S) > |S| = s$. Thus, $q \geqslant s + 1$.

Let $G_1, \ldots, G_q$ denote the odd components of $G \setminus S$. For $1 \leqslant i \leqslant q$, let $n_i = |V(G_i)|$, $e_i = |e(G_i)|$ and $t_i = e(G_i, S)$. Then $t_i \geqslant 1$ and $2e_i = kn_i - t_i$ for $1 \leqslant i \leqslant q$. Because each n_i is odd, it follows that k and t_i have the same parity for each i. Also, $t_1 + \cdots + t_q \leqslant e(S, G \setminus S) \leqslant ks$.

The previous inequality implies that $t_i < k$ for at least two i's, for $i = 1, 2$ say. Since t_i has the same parity as k, this implies $t_i \leqslant k - 2$ and so $n_i(n_i - 1) \geqslant 2e_i = kn_i - t_i \geqslant kn_i - k + 2$ for $i = 1, 2$. Therefore $n_i \geqslant k + \frac{2}{n_i - 1}$ and so $n_i \geqslant k + 1$ for $i = 1, 2$.

As in the proof of Theorem 11, it follows that for $i = 1, 2$,

$$\lambda_1(G_i) \geqslant \frac{2e_i}{n_i} + \frac{1}{2n_i k} = k - \frac{t_i}{n_i} + \frac{1}{2kn_i} \geqslant k - \frac{k-2}{k+1} + \frac{1}{(k+1)(k+2)}.$$

Finally, by interlacing the eigenvalues of $G_1 \cup G_2$ and G, it follows that $\lambda_2(G) \geqslant k - \frac{k-2}{k+1} + \frac{1}{(k+1)(k+2)} = k - 1 + \frac{3}{k+1} + \frac{1}{(k+1)(k+2)}$, a contradiction. □

We conclude with a modification of the construction of Brouwer and Haemers mentioned earlier. The modification shows that the bound in Theorem 13 is best up to order $O(k^{-2})$.

Let G' denote the graph obtained from K_{k+1}, k even, by deleting a matching of order $\frac{k-2}{2}$. Take $k - 1$ disjoint copies of G' together with a set N of $k - 2$ new vertices on which we select a perfect matching. Join each of the new vertices to one of the vertices of degree $k - 1$ in each copy of G'. Let H denote the graph obtained. Then H is k-regular on $(k-1)(k+1) + k - 2 = k^2 + k - 3$ vertices, an odd number (see Fig. 2, for the case $k = 4$). Also, $|\operatorname{odd}(H \setminus N)| = k - 1 > |N|$, so by Gallai's condition (15), H is not factor critical. However, it can be shown that

$$\lambda_2(H) = \frac{k - 2 + \sqrt{k^2 + 12}}{2} = k - 1 + \frac{3}{k+1} + O(k^{-2}).$$

Thus, there are k-regular graphs with λ_2 very close to $k - \frac{k-2}{k+1} + \frac{1}{(k+1)(k+2)}$ that are not factor-critical.

For example, when $k = 4$, Theorem 13 implies that any 4-regular graph with $\lambda_2 < 4 - \frac{4-2}{4+1} + \frac{1}{5 \cdot 6} < 3.6334$ is factor-critical. On the other hand, when $k = 4$, the previous construction (shown in Fig. 2), gives a 4-regular graph which is not factor-critical and has $\lambda_2 = \frac{2+\sqrt{28}}{2} = 3.6458$.

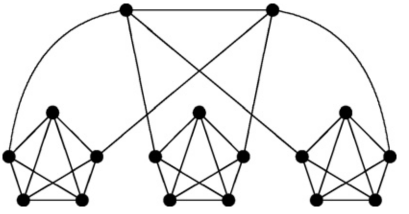

Fig. 2. A 4-regular graph which is not factor-critical. Here $\lambda_2 = 3.6458$.

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